p-HELSON SETS, $1 < p < 2$

BY

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ABSTRACT

A p-Helson set is defined to be a closed subset E of the circle group T with the property that every continuous function on E can be extended to the full circle in such a way that this extension has its sequence of Fourier coefficients in l^p. For $1 < p < 2$, the union of two such sets is again a p-Helson set. It is shown that the p-Helson sets ($p > 1$) differ from the Helson sets and also that the notion really depends on the index p. An analogue of H. Helson's result is given: a p-Helson set supports no nonzero measure with Fourier-Stieltjes transform in l^q, $1/p + 1/q = 1$.

1. Introduction

A closed subset E of the circle group T is called a Helson set if every continuous function on E is the restriction to E of a function with an absolutely convergent Fourier series. If $M(E)$ denotes the space of complex Borel measures carried by E, this is equivalent to the existence of a positive constant δ such that

$$\| \mu \|_M \leqq \delta \| \hat{\mu} \|_\infty, \qquad (\mu \in M(E)); \tag{1}$$

here $\| \mu \|_M = |\mu|(T)$ is the total variation norm of the measure μ, and $\| \hat{\mu} \|_\infty$ is the l^∞-norm of the Fourier-Stieltjes transform of μ. Condition (1) makes it plain why the Helson sets are thought of as the continuous analogues of the Sidon sets.

In [8], Rudin introduced a generalization of the Sidon sets, called $\Lambda(p)$-sets, or p-Sidon sets. In this paper, we will investigate one possible dualization of these p-Sidon sets. More precisely, if $1 \leqq p \leqq 2$, let $A^p(T)$ stand for the space of continuous complex functions on T whose sequence of Fourier coefficients lie in l^p; if every continuous function on E is the restriction to E of a function in $A^p(T)$, then we will say that E is a p-Helson set. When $p = 1$, we have the usual Helson

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sets, whereas when $p = 2$, every closed set satisfies the requirement since every continuous function on T has a square summable sequence of Fourier coefficients. We will therefore confine our attention to the case in which $1 < p < 2$.

Here is an outline of the results. We first give the appropriate analogue of the condition (1) for p-Helson sets. Using it, we prove that a p-Helson set cannot carry a nonzero measure whose Fourier-Stieltjes transform belongs to l^q, where $q = p/(p-1)$. With a bit more work, we find a characterization of p-Helson which involves the sets of zero measure with respect to a certain subspace of $M(T)$. Finally, we use our necessary and sufficient condition to give examples of p-Helson sets that are not Helson sets, and examples of p_2-Helson sets that are not p_1-Helson sets, where $1 < p_1 < p_2 < 2$.

2. $A^p(T)$ and its dual

We norm the space $A^p(T)$ by

$$\|f\| = \max\{\|f\|_\infty, \|\hat{f}\|_{l^p}\} \qquad (f \in A^p(T)). \tag{2}$$

It is easy to see that $A^p(T)$ is then a Banach space. We need to find the dual space of $A^p(T)$ under this norm. To do this, observe that under the sup-norm ($\|f\|_\infty$), $A^p(T)$ is dense in $C(T)$, and that under the l^p-norm ($\|\hat{f}\|_{l^p}$), $A^p(T)$ is dense in l^p. Therefore, the duals of $A^p(T)$ with respect to these two norms are $M(T)$ and l^q, respectively. One can then see that the dual of $A^p(T)$ under the norm (2) can be identified with the product $M(T) \times l^q$, factored by the subspace N of all pairs $(\mu, \hat{\mu})$ with $\mu \in M_q(T)$, where $M_q(T)$ is the set of all μ in $M(T)$ with $\hat{\mu}$ in l^q. That is to say, if Λ is a continuous linear functional on $A^p(T)$, then there exist μ in $M(T)$ and $S = \{d_n\}_{-\infty}^{+\infty}$ in l^q with

$$\Lambda f = \int f \, d\mu - \sum_{-\infty}^{+\infty} \hat{f}(n) \cdot d_n \qquad (f \in A^p(T))$$

and

$$\|\Lambda\| = \inf\{\|\mu - \lambda\|_M + \|S - \hat{\lambda}\|_{l^q} : \lambda \in M_q(T)\}.$$

We will abbreviate this last infimum by $\||(\mu, S)|\|$, and if $S \equiv 0$, we will write $\||\mu|\|$, instead of $\||(\mu, 0)|\|$.

We can interpret the collection of restrictions of the functions in $A^p(T)$ to a fixed closed set E as a quotient space of $A^p(T)$ in the usual way. If we let I_E^p be the set of all functions in $A^p(T)$ which vanish on E, then the quotient of $A^p(T)$ by

I_E^p, which we denote by $A^p(E)$, is naturally isomorphic to this set of restrictions. Hence, E is a p-Helson set if and only if $A^p(E) = C(E)$.

We can now give the promised analogue of (1).

THEOREM 1. *Let E be a closed subset of T, and $1 \leqq p \leqq 2$. E is a p-Helson set if and only if there is a positive number δ such that for each $\mu \in M(E)$,*

$$\| \mu \|_M \leqq \delta \; \| | \mu | \| . \tag{3}$$

PROOF. The linear map $S: A^p(E) \to C(E)$ defined by $S([f]) = f|_E$ is well-defined, one-to-one and continuous. Clearly, E is a p-Helson set if and only if S is onto $C(E)$. By a classical theorem [5, p. 141], S is onto provided its adjoint S^* has a continuous inverse. Now, the adjoint $S^*: M(E) \to A^p(E)^*$ is given by $S^*\mu = (\mu, 0)$ for every μ in $M(E)$. Therefore, (3) is seen to be the statement that S^* has a continuous inverse. Conversely, if $A^p(E) = C(E)$, then the adjoint S^* is onto $A^p(E)^*$, is one-to-one and continuous. Thus, by the Open Mapping Theorem, there is a $\delta > 0$ for which (3) holds. □

3. Measures carried by p-Helson sets

From now on, we will take p with $1 < p < 2$, and let q be its conjugate exponent, $q = p/(p-1)$. We want to investigate the consequences of the following condition on a closed set E: There is a positive constant δ such that

$$\| \mu \|_M \leqq \delta \cdot \| \hat{\mu} \|_{l^q} \qquad (\mu \in M_q(E)), \tag{4}$$

where $M_q(E) = M_q(T) \cap M(E)$. This condition must be satisfied if E is p-Helson, because of Theorem 1 and the simple fact that $\| | \mu | \| \leqq \| \hat{\mu} \|_{l^q}$, if $\mu \in M_q(T)$. We will show that (4) cannot hold for any set E unless $M_q(E)$ is zero.

THEOREM 2. *Let E be a closed subset of T and $2 < q < \infty$. If $M_q(E)$ is nonzero, then*

$$\sup \left\{ \frac{\| \mu \|_M}{\| \hat{\mu} \|_{l^q}} : \mu \in M_q(E), \; \mu \neq 0 \right\} = +\infty.$$

For the proof, we need the following lemmas, the first of which is proved in [5, p. 143].

LEMMA 1. *Let $c_1, \ldots, c_k$ be given complex numbers, and consider the random variable*

$$\pm c_1 \pm \cdots \pm c_k \, .$$

where the $\pm$*'s are chosen randomly, each with probability* $\frac{1}{2}$*, and independently of one another. If* E *denotes the expected value, then*

$$E(|\pm c_1 \pm \dots \pm c_k|) \geqq \frac{1}{\sqrt{3}}(|c_1|^2 + \dots + |c_k|^2)^{\frac{1}{2}}.$$

LEMMA 2. *Let* γ *be in* $M_q(T)$, $2 < q < \infty$. *To each* $k = 2, 3, 4, \dots$, *there corresponds a positive integer* N *such that whenever* $n_1, \dots, n_k$ *are integers which satisfy* $|n_i - n_j| > 2N$ *when* $i \neq j$, *then for every choice of* $\pm$, *the Fourier-Stieltjes transform of the measure*

$$d\gamma_k(t) = \left(\sum_{j=1}^{k} \pm e^{in_jt}\right) d\gamma(t)$$

satisfies $\|\hat{\gamma}_k\|_{l^q} \leqq c \cdot k^{1/q} \cdot \|\hat{\gamma}\|_{l^q}$, *where* $c = 2$.

PROOF. Let $\gamma \in M_q(T)$, and k be given. Choose N so large that

$$\left(\sum_{|n|>N} |\hat{\gamma}(n)|^q\right)^{1/q} \leqq k^{1/q-1} \|\hat{\gamma}\|_{l^q}. \tag{5}$$

Let the integers $n_1, \dots, n_k$ be given with $|n_i - n_j| > 2N$, if $i \neq j$. Put

$$d\gamma^0(t) = \left(\sum_{n=-N}^{N} \hat{\gamma}(n)\, e^{int}\right) dt$$

and $d\gamma' = d\gamma - d\gamma^0$. Then

$$\begin{aligned} d\gamma_k(t) &= \left(\sum_{j=1}^{k} \pm e^{in_jt}\right) d\gamma^0(t) + \left(\sum_{j=1}^{k} \pm e^{in_jt}\right) d\gamma'(t) \\ &= d\gamma_k^0(t) + d\gamma_k'(t). \end{aligned}$$

Because of the gaps between the integers $n_1, \dots, n_k$, every Fourier-Stieltjes coefficient of $d\gamma_k^0$ is a coefficient of $d\gamma^0$ (up to sign) and each coefficient of $d\gamma^0$ appears (up to sign) precisely k times in the sequence $\{\hat{\gamma}_k^0(n)\}_{n=-\infty}^{+\infty}$. Therefore $\|\hat{\gamma}_k^0\|_{l^q} \leqq k^{1/q} \|\hat{\gamma}^0\|_{l^q}$. On the other hand

$$\|\hat{\gamma}_k'\|_{l^q} \leqq k \|\hat{\gamma}'\|_{l^q} \leqq k^{1/q} \|\hat{\gamma}\|_{l^q},$$

by (5). The conclusion follows. □

PROOF OF THEOREM 2. Pick $\mu \neq 0$ from $M_q(E)$. We will exhibit a sequence $\{\nu_k\}_{k=2}^{\infty}$ of (non-zero) measures in $M_q(E)$ for which

$$\|\nu_k\|_M \geqq c \cdot k^{\frac{1}{2}} \cdot \|\mu\|_M, \tag{6}$$

and

(7) $$\| \hat{\nu}_k \|_{l^q} \geqq c' \cdot k^{1/q} \cdot \| \hat{\mu} \|_{l^q},$$

where the constants c, c' are independent of k.

So let an integer $k \geqq 2$ be given. Apply Lemma 2 to the measure μ to find an integer N so that the conclusion of that lemma holds. Choose integers $n_1, \dots, n_k$ in such a way that $|n_i - n_j| > 2N$ if $i \neq j$. Then, for every choice of $\pm$, the Fourier-Stieltjes transform of the random measure

$$d\nu_k(t) = \left(\sum_{j=1}^{k} \pm e^{in_j t} \right) d\mu(t)$$

satisfies (7). By Lemma 1, the expected value of the total variation of ν_k is

$$\begin{aligned} \mathrm{E}(\| \nu_k \|_M) &= \mathrm{E}\left(\int_E d|\nu_k| \right) \\ &= \int \mathrm{E}\,(| \pm e^{in_1 t} \pm \dots \pm e^{in_k t}|) d|\mu|(t) \\ &\geqq \frac{1}{\sqrt{3}} \cdot k^{\frac{1}{2}} \cdot \| \mu \|_M. \end{aligned}$$

Therefore, there is at least one choice of $\pm$ for which (6) holds, with $c = 1/\sqrt{3}$. Since (7) holds for all choices, we are done. □

As we pointed out at the beginning of this section, Theorem 2 has the following corollary, which we may consider as the analogue of Helson's Theorem: If E is a Helson set and if $\mu \in M(E)$ has $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, then $\mu = 0$ [3].

COROLLARY. *If E is a p-Helson set, and $1 < p < 2$, then*

$$M_q(E) = \{0\}, \text{ where } q = p/(p-1).$$

4. Necessary and sufficient conditions

It is easy to see that $\| |\mu| \| = \| \mu \|_M$ provided every λ in $M_q(T)$ satisfies $|\lambda|(E) = 0$. Thus, Theorem 1 shows that E is a p-Helson set whenever each λ in $M_q(T)$ has none of its mass on E. This turns out to be a necessary condition also. To see why, we first prove an extension of the corollary to Theorem 2.

THEOREM 3. *If E is p-Helson, where $1 < p < 2$, then $\overline{M_q(T)} \cap M(E) = \{0\}$, the closure taking place in the total variation norm of $M(T)$.*

Note that Theorem 3 would follow from the corollary if it were true that the restriction of an M_q-measure to a closed set was still an M_q-measure. Nevertheless, we can establish Theorem 3 using an argument similar to the one in the proof of Theorem 2, together with the following lemma.

LEMMA 3. *Let* μ *in* $M_0(T) \equiv \{\mu \in M(T): \hat{\mu} \in c_0\}$ *be given. To each* $k = 2,3,4,\ldots$, *there corresponds an integer* $M > 0$ *such that whenever* $n_1,\ldots,n_k$ *are integers which satisfy* $|n_i - n_j| > M$ *for* $i \neq j$, *then for every choice of* $\pm$, *the total variation of the measure*

$$d\mu_k(t) = \left(\sum_{j=1}^{k} \pm e^{in_j t}\right) d\mu(t)$$

satisfies $\|\mu_k\|_M \leqq B \cdot k^{\frac{1}{2}} \cdot \|\mu\|_M$, *where* $B = \sqrt{2}$.

PROOF. Let μ in $M_0(T)$ be given and fix $k \in \{2,3,4,\cdots\}$. Find M so large that if $|m| > M$, then

$$||\hat{\mu}|(m)| \leqq \frac{\|\mu\|_M}{k(k-1)}. \tag{8}$$

This can be done because $\mu \in M_0(T)$ if and only if $|\mu| \in M_0(T)$. Now assume $n_1,\cdots,n_k$ are integers with $|n_i - n_j| > M$ when $i \neq j$. The convexity of $t^2 (t > 0)$ and Jensen's inequality [10, p. 61] show that the square of the total variation of μ_k is no larger than the total variation of μ times

$$\begin{aligned} &\int \left|\sum_{j=1}^{k} \pm e^{in_j t}\right|^2 d|\mu|(t) \\ &= \int \left(\sum_{j=1}^{k} \pm e^{in_j t}\right)\left(\sum_{p=1}^{k} \pm e^{-in_p t}\right) d|\mu|(t) \\ &= k \cdot \|\mu\|_M + \sum_{\substack{j,p=1 \\ j \neq p}}^{k} \pm |\hat{\mu}|(n_p - n_j) \\ &\leqq k \cdot \|\mu\|_M + \|\mu\|_M; \end{aligned}$$

This last inequality is true because of (8). Hence,

$$\|\mu_k\|_M^2 \leqq k \cdot \|\mu\|_M^2 + \|\mu\|_M^2,$$

and the lemma follows. □

PROOF OF THEOREM 3. Assume that $\overline{M_q(T)} \cap M(E)$ is nonzero; say $\mu \neq 0$ is in $M(E)$ and measures $\gamma_k \in M_q(T)$ can be found with $\|\mu - \gamma_k\|_M \to 0$ as $k \to \infty$. It is easy to see that we may assume the Fourier-Stieltjes transforms $\hat{\gamma}_k$ satisfy

$$k^{1/q - \frac{1}{2}} \|\hat{\gamma}_k\|_{l^q} \to 0 \text{ as } k \to \infty,$$

because $q > 2$. To show that E is not a p-Helson set, we will exhibit a sequence of nonzero measures μ_k in $M(E)$ for which $|||\mu_k|||/\|\mu_k\|_M \to 0$ as $k \to \infty$, and then appeal to Theorem 1.

First observe that the measure μ has $\hat{\mu} \in c_0$; this follows from the fact that each γ_k is in $M_q(T) \subset M_0(T)$ and $\|\hat{\mu} - \hat{\gamma}_k\|_{l^\infty} \leqq \|\mu - \gamma_k\|_M$. Hence, $\mu - \gamma_k$ is in $M_0(T)$, for all k. Let $k \in \{2, 3, 4, \ldots\}$ be fixed. By Lemma 3 (applied to the measure $\mu - \gamma_k$), there is an $M > 0$ such that whenever $n_1, \cdots, n_k$ are integers with $|n_i - n_j| > M$ for $i \neq j$, then the total variation of

$$d\nu_k(t) = \left(\sum_{j=1}^{k} \pm e^{in_jt}\right) d(\mu - \gamma_k)(t)$$

satisfies

$$\|\nu_k\|_M \leqq B \cdot k^{\frac{1}{2}} \cdot \|\mu - \gamma_k\|_M \tag{9}$$

for every choice of $\pm$. By Lemma 2 (applied to γ_k), there is an $N > 0$ such that whenever $n_1, \ldots, n_k$ are integers with $|n_i - n_j| > 2N$ for $i \neq j$, then the Fourier-Stieltjes transform of the measure

$$d\rho_k(t) = \left(\sum_{j=1}^{k} \pm e^{in_jt}\right) d\gamma_k(t)$$

satisfies

$$\|\hat{\rho}_k\|_{l^q} \leqq c \cdot k^{1/q} \cdot \|\hat{\gamma}_k\|_{l^q} \tag{10}$$

for every choice of $\pm$.

Therefore, if we choose integers $n_1, \ldots, n_k$ so that $|n_i - n_j| > 2\max\{M, N\}$ for $i \neq j$, then both (9) and (10) will hold for every choice of $\pm$. Let the integers $n_1, \ldots, n_k$ be determined by such a choice.

By Lemma 1, if μ_k in $M(E)$ is the random measure

$$d\mu_k(t) = \left(\sum_{j=1}^{k} \pm e^{in_jt}\right) d\mu(t)$$

then, as in the proof of Theorem 2, we have

$$\mathrm{E}(\|\mu_k\|_M) \geqq \frac{1}{\sqrt{3}} \cdot k^{\frac{1}{2}} \cdot \|\mu\|_M.$$

Therefore, there is at least one choice of $\pm$ for which

$$\|\mu_k\|_M \geqq \frac{1}{\sqrt{3}} \cdot k^{\frac{1}{2}} \cdot \|\mu\|_M \tag{11}$$

holds. Let the measures μ_k, ρ_k (and hence ν_k) be determined by such a choice of $\pm$. Then, because (9), (10) and (11) hold, we have

$$\frac{\||\mu_k|\|}{\|\mu_k\|_M} \leqq \frac{\|\mu_k - \rho_k\|_M}{\|\mu_k\|_M} + \frac{\|\hat{\rho}_k\|_{l^q}}{\|\mu_k\|_M} = \frac{\|\nu_k\|_M}{\|\mu_k\|_M} + \frac{\|\hat{\rho}_k\|_{l^q}}{\|\mu_k\|_M}$$
$$\leqq \frac{\sqrt{3}\cdot B \cdot \|\mu - \gamma_k\|_M}{\|\mu\|_M} + \frac{\sqrt{3}\cdot c \cdot k^{1/q-\frac{1}{2}}\|\hat{\gamma}_k\|_{l^q}}{\|\mu\|_M},$$

and both of these quantities tend to zero as $k \to \infty$. Therefore, by Theorem 1, E is not a p-Helson set. □

We now gather our equivalences for a p-Helson set:

THEOREM 4. *Let E be a closed subset of T, $1 < p < 2$, $q = p/(p-1)$. The following are equivalent:*

(a) *E is a p-Helson set;*

(b) *there is a $\delta > 0$ such that $\|\mu\|_M \leqq \delta \cdot \||\mu|\|$, for all $\mu \in M(E)$;*

(c) $\overline{M_q(T)} \cap M(E) = \{0\}$;

(d) *$|\gamma|(E) = 0$, for every $\gamma \in M_q(T)$;*

(e) *$\||\mu|\| = \|\mu\|_M$, for every $\mu \in M(E)$.*

PROOF. Theorem 1 established the equivalence of (a) and (b). Theorem 3 showed that (b) implies (c). Since the implications (d) → (e) → (b) are both evident, it only remains to be shown that (c) → (d). Suppose that $|\gamma|(E) > 0$ for some $\gamma \in M_q(T)$, and put $d\gamma_k = f_k d\gamma$, where $f_k \in A(T)$, $0 \leqq f_k \leqq 1$, $f_k = 1$ on E and $f_k \to \chi_E$, pointwise almost everywhere with respect to $|\gamma|$; this can be done because $A(T)$ is a normal family of functions on T [4, p. 341]. Hence,

$$\|\chi_E d\gamma - \gamma_k\|_M = \int |\chi_E - f_k| d|\gamma| \to 0.$$

This shows that the measure $\chi_E d\gamma$ belongs to the closure of $M_q(T)$; since it is nonzero and carried by the set E, we are done. □

COROLLARY. *If $E_1, E_2, \cdots$ are p-Helson sets, $1 < p < 2$, then so is their union, provided it is closed. In particular, the union of two p-Helson sets is again a p-Helson set.*

PROOF. If $\gamma \in M_q(T)$, then $|\gamma|$ annihilates each E_n, so it annihilates the union. By (d) of Theorem 4, the union is p-Helson. □

Before proceeding with some examples, we want to point out a few other simple consequences of Theorem 4. First, (d) shows that p-Helson sets $(1 < p < 2)$ have zero Lebesgue measure. This fact seems to have been known; it appears as an exercise in one of the standard works on trigonometric series [1, p. 359]. It also shows that not all closed sets of measure zero are p-Helson, for Kahane and Salem [5, p. 106] construct a perfect set of Hausdorff dimension α where $2/q < \alpha < 1$, which supports a nonzero measure from $M_q(T)$. Finally, we remark that Varopoulos [12] has recently established that the Corollary of Theorem 4 is also true when $p = 1$.

5. Some examples

By Theorem 4(d), a closed set E is p-Helson if and only if each λ in $M_q(T)$ has none of its mass on E. The simplest class of sets with this property are the U_0-sets, that is to say, the closed sets E such that no nonzero measure carried by E has its Fourier-Stieltjes transform vanishing at infinity. For example, every countable set is U_0, since a measure μ with $\hat{\mu} \in c_0$ must be continuous: $\mu(\{x\}) = 0$ for each x. [9, p. 118]. To see that U_0-sets are p-Helson ($p > 1$), we need the following lemma.

LEMMA 4. *If $\mu \in M_0(T)$ then $h\mu \in M_0(T)$ for any bounded measurable function h, since h is the limit of trigonometric polynomials in $L^1(|\mu|)$.*

THEOREM 5. *If the closed set E is a U_0-set, then E is a p-Helson set, for every $p > 1$.*

PROOF. If $\lambda \in M_q(T)$, $2 < q < \infty$, then $\chi_E d\lambda$ is in $M_0(T)$, by the lemma. Since $\chi_E d\lambda$ is carried by the U_0-set E, it must be the zero measure, that is, $|\lambda|(E) = 0$. □

We point out that U_0-sets need not be Helson sets. In fact, there are plenty of countable sets that are not Helson sets; see [6, p. 32] for a very general construction.

COROLLARY. *If E is a symmetrical perfect set with constant ratio ξ, with $1/\xi$ a Pisot number, then E is a p-Helson set for every $p > 1$; furthermore, no such set is a Helson set.*

This is true because Salem and Zygmund have shown, see [5, p. 74], that such a set is U_0. A theorem of Kahane and Salem, see [6, p. 32] shows that no symmetrical set (constant ratio or not) can be a Helson set. We point out that there do exist perfect symmetrical sets with variable ratios which are U_0-sets, and hence p-Helson, $p > 1$.

We now want to show that there are sets E which are not U_0, but which are still p-Helson for every $p > 1$. This will be obtained as a corollary of the following theorem.

THEOREM 6. *Let E be a closed set such that the m-fold sum $E + \cdots + E$ has*

Lebesgue measure zero. Then E is a p-Helson set for every $p > 1$ whose conjugate exponent q satisfies $q \leqq 2m$, that is, for any $p \geqq 2m/2m - 1$.

PROOF. Let $\gamma \in M_q(T)$, where $q \leqq 2m$. Let $\nu = \chi_E d\gamma$, and consider the m-fold convolution $\nu^m = \nu * \cdots * \nu$. Since ν is carried by E, ν^m is carried by the m-fold sum $E + \cdots + E$, and hence is singular, by the hypothesis.

We claim that ν^m is also absolutely continuous. To see this, first observe that the m-fold convolution $\gamma^m = \gamma * \cdots * \gamma$ has a square-summable Fourier-Stieltjes transform, since $2m \geqq q$. Consequently, γ^m is absolutely continuous. Now the measure ν need not belong to $M_q(T)$, but it does belong to $\overline{M_q(T)}$, the closure in total variation norm. To see this, we can use an argument similar to that in the proof of Theorem 4: approximate χ_E pointwise boundedly by a sequence of functions $g_n \in A(T)$, and observe that the measures $g_n d\gamma$ (which are still in $M_q(T)$) converge in total variation to $\chi_E d\gamma$. Now it is a simple matter to show that because $\chi_E d\gamma$ belongs to $\overline{M_q(T)}$, its m-fold convolution power must be absolutely continuous, since if measures ν_n converge to ν (in variation), then the m-fold convolutions ν_n^m converge to ν^m (in variation). Thus, ν^m is absolutely continuous.

Therefore, ν^m must be the zero measure. But ν^m cannot be the zero measure unless $d\nu = \chi_E d\gamma$ is the zero measure. Consequently, $|\gamma|(E) = 0$ and the proof is complete. □

COROLLARY 1. *A closed independent set E is p-Helson for every $p > 1$.*

PROOF. By an independent set we mean the following: If $x_1, \cdots, x_n$ are distinct points of E, and $n_1, \cdots, n_k$ are integers with $n_1 x_1 + \cdots + n_k x_k = 0$, then $n_1 = \cdots = n_k = 0$. To prove the corollary, it suffices to show that the Lebesgue measure of $E + \cdots + E$ (any number of summands) is zero. In fact, it is an easy exercise to show, using the fact that proper measurable subgroups of T have measure zero [7, p. 8], that $Gp(E)$, the subgroup of T generated by E, has measure zero. For an even stronger result, see [2].

To complete the proof, observe that the hypothesis of Theorem 6 is satisfied for every positive integer m. □

We can now see that the converse of Theorem 5 is not true; it is well known that there exist perfect independent sets E with $M_0(E) \neq \{0\}$; see [5, p. 106].

As our final class of examples, we take certain symmetrical perfect sets with variable ratios; for a description, see [5, ch. 1].

COROLLARY 2. *If $E = E(\xi_k)$ is the symmetrical perfect set with variable ratios*

ξ_k, *and* $\liminf_{k\to\infty} 2^{mk}\xi_1 \cdots \xi_k = 0$, *for some positive integer m, then E is a p-Helson set for every* $p \geqq 2m/2m-1$.

It is an easy matter to see that the Lebesgue measure of $E + \cdots + E$ (m summands) is no larger than

$$m \cdot \liminf_{k\to\infty} 2^{mk}\xi_1 \cdots \xi_k;$$

hence Theorem 6 applies.

COROLLARY 3. *If* $E = E(\xi_k)$ *and* $\xi_k \to 0$, *then E is p-Helson for every* $p > 1$.

Indeed, the hypothesis of Corollary 2 is satisfied for every positive integer m. Corollary 3 furnishes further examples of p-Helson sets for every $p > 1$ which are not U_0-sets, because a theorem of Kahane and Salem [5, p. 103] shows that there are many symmetrical perfect sets E with ratios tending to zero such that $M_0(E) \neq \{0\}$.

6. Dependence on p

We finally point out that the notion of a p-Helson set really does depend on p. We will use a theorem of R. Salem, together with our results.

THEOREM (*Salem*, [11]). *Consider all the symmetrical perfect sets* $E = E(\xi_k)$ *for which* $a_k \leqq \xi_k \leqq b_k$, *where* $0 < a_k < b_k < \frac{1}{2}$, *and where the sequences* $\{a_k\}$, $\{b_k\}$ *satisfy the following*:

(i) $b_k - a_k \geqq 1/\omega(k)$, *where* $\omega(k)$ *is a positive nondecreasing sequence such that* $\log \omega(k) = o(k)$ *as* $k \to \infty$;

(ii) $\liminf_{k\to\infty} (a_1 \cdots a_k)^{1/k} = \alpha > 0$;

then, there is a constant q_0, *depending only on* α, *such that for every* $q > q_0$, *the series*

$$\sum_{-\infty}^{+\infty} |\hat{\mu}(n)|^q$$

converges for almost all sets E, where $\hat{\mu}(n)$ *is the nth Fourier-Stieltjes coefficient of the L-measure* μ *carried by E.*

Let us fix an integer $m \geqq 2$, and choose the sequences $\{a_k\}$ and $\{b_k\}$ in Salem's Theorem as follows:

$$a_k = (\tfrac{1}{4})^m, \; b_k = (\tfrac{1}{3})^m \text{ for all } k.$$

Then conditions (i) and (ii) of the theorem are satisfied. Therefore, almost all the sets E have $M_{q_1}(E) \neq \{0\}$ for $q_1 > q_0$. Consequently, almost all the sets E are not

p_1-Helson for p_1 with $1 < p_1 < q_0/(q_0 - 1)$, by Theorem 4. On the other hand, Corollary 2 of Theorem 6 shows that every set E admitted in Salem's Theorem is a p_2-Helson set, where $p_2 \geqq 2m/(2m-1)$, because, for any $E = E(\xi_k)$, the measure of the m-fold sum $E + \cdots + E$ is no larger than

$$\liminf_{k \to \infty} 2^{mk} b_1 \cdots b_k = 0.$$

We should remark that we have *not* proved that given arbitrary p_1, p_2 with $1 < p_1 < p_2 < 2$, there exists a p_2-Helson set which is not a p_1-Helson set. However, if we take the integer m (in our application of Salem's theorem above) to be very large, we do make $p_2 - p_1$ arbitrarily small, although then both p_1, p_2 will be very close to 1.

7. Some questions

We have not been able to determine whether $M_q(E) = \{0\}$ implies E is p-Helson. This is probably false, because the restriction of an M_q measure to a closed set does not seem to be an M_q-measure.

Also, one can replace the space $A^p(T)$ by the subspace of "analytic" functions in $A^p(T)$, that is, let $A_+^p(T)$ be the collection of all functions in $A^p(T)$ whose negative Fourier coefficients vanish. Define $A_+^p(E)$ in the usual way, and call a closed set $E \subset T$ a p-Carleson set if $A_+^p(E) = C(E)$. Wik [13] showed that the Carleson sets (i.e., our 1-Carleson sets) are no different than the Helson sets. We have been unable to verify this when $1 < p < 2$, although it is easily seen to be false when $p = 2$: a 2-Carleson set must have Lebesgue measure zero, whereas every closed set is 2-Helson. On the other hand, all of our theorems concerning p-Helson sets have valid analogues for p-Carleson sets. In particular, every example of a p-Helson set that we have given, can also be shown to be a p-Carleson set, and Salem's theorem can again be used to see that p-Carleson does depend on p.

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